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Weyl series for Aharonov–Bohm billiards

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Abstract

Following a conjecture of Berry and Howls (1994) concerning the geometric information contained within the high orders of Weyl series, we examine such series for the average spectral properties of two- and three-dimensional quantum ball billiards threaded by a single flux line at the centre. We adapt a Mellin-based scheme of Bordag *et al* (1996) to generate the Weyl series. It is shown that for a circular billiard, only a single Weyl series term is changed and thus the flux line only induces a simple constant shift in the *average* properties of the spectrum, although the fluctuations about this average will still be flux dependent. This implies that the late terms in the expansion are dominated by the diametrical periodic orbit of the unfluxed circle, rather than the shorter diffractive orbits encountering both the billiard boundary and the flux line. For a spherical billiard with flux the late terms suffer modifications which can be linked to diffractive orbits. The origins of the differences between the structure of the series are traced to the interaction of the geometry and symmetry breaking.

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1. Introduction

The higher orders of large energy expansions of spectral functions have been the subject of some recent study. Various approaches have been taken towards different goals.

A sequence of papers starting with the theory of Bordag *et al* (1996a) provided technically detailed methods for generating the terms in the asymptotic expansion of spectral functions of systems with known circular or spherically symmetric eigenfunctions. The method is based on a careful analytic continuation of an integral representation of spectral zeta functions leading to residue calculation of asymptotic expansions. Applications and extensions have involved the variation of boundary conditions (Dowker 1996, Dowker *et al* 1996, Bordag and Vassilevic 1999), the use on different metrics (Dowker and Kirsten 2001) and the calculation of Casimir energies (Bordag *et al* 1997, Elizalde *et al* 1998). Levitin (1998) derived a method which generates the heat invariants of Euclidean balls in arbitrary dimensions. Others have considered the effect of uniform magnetic fields on the asymptotics of the heat kernel (Narevich *et al* 1998, Spehner *et al* 1998, Narevich and Spehner 1999).

Much of the above work has been concerned with the derivation of the coefficients *per se*. Another vein has considered the 'asymptotics of the asymptotics' and has sought to determine whether global properties of the classical system are reflected within the higher order coefficients, mirroring the geometric information (area, perimeter, connectivity etc) contained in the first few terms (Kac 1966). Berry and Howls (1994) (hereafter called BH) conjectured that the higher order terms of the Weyl series expansion for the resolvent of 2D quantum billiards obeying Helmholtz' equation with Dirichlet boundary data

$$g(s) = \lim_{N\to\infty}\left[\sum_{n=1}^{N}\frac{1}{E_n+s^2} - \frac{A}{4\pi}\log\left(\frac{E_n}{s^2}\right)\right] \tag{1}$$

(and by appropriate integral transformation, for other spectral functions) behaved as a factorial over a power of a periodic orbit associated with the dynamics of the corresponding classical system. This conjecture was examined subsequently for billiards with corners (Howls and Trasler 1998) and in different dimensions (Howls and Trasler 1999) (hereafter called HT) where systematic corrections to this asymptotic behaviour were conjectured:

$$g(s) \sim \sum_{r=1}^{\infty}\frac{c_r}{s^r} \qquad c_r \sim \sum_{l_{\mathrm{p.o.}}}\sum_{p=0}^{\infty} a_{r,p,\mathrm{p.o.}}\frac{\Gamma(r-p+\mu_p)}{l_{\mathrm{p.o.}}^{r-p+\mu_p}} \tag{2}$$

where $l_{\mathrm{p.o.}}$ are associated periodic orbits of the classical system. These orbits may be periodic orbits of the system in question, or of one related by symmetry (BH). Variation of boundary data was also considered by Trasler (1998). The conclusion of these works was that it is not always the shortest periodic orbit that dominates the late term behaviour and that the presence of corners, holes and the dimensionality of the billiard can all affect the result.

In this paper we combine the above strands to investigate the high orders in the Weyl expansion of two Aharonov–Bohm billiards (Aharonov and Bohm 1959). These are billiard systems (Berry and Robnik 1986) where a point particle of mass m and charge q encounters a magnetic vector potential $\boldsymbol{A}$ so that the quantum eigenfunctions ψ and eigenenergies E satisfy the modified equation

$$\frac{1}{2m}(-\mathrm{i}\hbar\nabla - q\boldsymbol{A})^2\psi = E\psi \tag{3}$$

with Dirichlet boundary conditions. The magnetic vector potential $\boldsymbol{A}$ is any function that gives rise to a flux

$$\boldsymbol{B} \equiv \nabla\times\boldsymbol{A} = 2\pi\delta(\boldsymbol{r}-\boldsymbol{r}_0). \tag{4}$$

In what follows we shall use α to denote the scaled flux

$$\alpha \equiv q\Phi/h \qquad \Phi = \oint_{r_0}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{r}. \tag{5}$$

There are several reasons for studying this type of system. First, Berry (1986) derived the flux correction to c_2 of the spectral counting function for arbitrary 2D billiards using the free Green function

$$c_2 = \frac{1}{6} - \frac{\alpha(1-\alpha)}{2}. \tag{6}$$

A justification of his calculation for the resolvent of general smooth billiards in $D = 2$ appears in appendix A. Here we seek to study the effect of the boundary on the higher order $c_r(r > 2)$ in the Weyl series, both in $D = 2$ and 3. Secondly, the flux line introduces diffractive orbit contributions to the oscillatory fluctuations in the semiclassical trace formula (Sieber 1999, Riemann *et al* 1996). The classical actions of periodic orbits enclosing the flux line acquire an additional flux-dependent term (although their spatial length remains unchanged). We seek to determine the influences of these additional geometric properties (if any) on the asymptotic form of the high orders. Thirdly, the classical dynamics of the fluxed system are still underpinned by periodic orbits but the time-reversal and spatial symmetries are broken by the flux-line. We seek to discover if the resurgence properties of the modified Weyl series can be seen to generate the symmetry-breaking prefactors of the periodic orbit fluctuations (Reimann *et al* 1996, Tanaka *et al* 1996, Sieber 1999). In addition the Aharonov–Bohm billiards provide a test of the late-term conjecture of BH for a more general quantum operator.

We choose to study two systems. The first is a $D = 2$ circular billiard with a flux line at the origin and Dirichlet boundary conditions, which breaks the time reversal symmetry. For the other we thread a flux line between the geographical poles in the interior of a $D = 3$ ball of radius R and apply Dirichlet conditions on the (inner) surface. This breaks the $SO(3)$ symmetry of the spherical billiard reducing it to a $U(1)$ axial symmetry. Physically both systems acquire diffractive orbits which scatter from the flux line and bounce on the boundary. The contributions of these orbits to the oscillatory corrections to the Weyl series of the fluxed circle have been studied by Reimann *et al* (1996) and Sieber (1999). The circle and sphere have been chosen for their simplicity, but also to test how the symmetry breaking interacts with the dimensional differences observed in HT, whereby the Weyl series of odd-dimensional balls are dominated by the triangular orbits, rather than the shorter diametrical orbits observed in even dimensions.

Note that the statistical distribution of the eigenvalues of more general 2D flux-dependent billiards have been discussed previously (Berry and Robnik 1986). The symmetry-breaking effect of a uniform magnetic field on a spherical quantum billiard has been discussed by Tanaka *et al* (1996) in the context of electronic 'supershells' for metallic clusters. The superposition of an Aharonov–Bohm solenoid on a magnetic monopole at the origin in a spherical polar geometry has been studied by Yeo and Moore (1998) as a model of a thin film superconductor.

It is possible to use the Green function techniques of BH and HT to generate the Weyl series for the fluxed billiards. However, each term in the Weyl series is generated from a process requiring several levels of formal expansions (cf HT), complicated further by flux-dependent oscillatory contributions that nevertheless sum conditionally to algebraic orders. To avoid such complications, we generalise a method due to Bordag, Elizalde and Kirsten (1996), hereafter called BEK. This approach seeks to analytically continue zeta function representations to the negative axis before relating them to the Weyl coefficients and is explained in sections 2 and 3. The method is valid for radially and spherically symmetric systems where the radial eigenfunctions are Bessel functions. We generalise the method to allow flux contributions in the $D = 2$ circle in section 4. Section 5 deals with the fluxed $D = 3$ ball. The Green function approach can be used to check and explain the asymptotic behaviour of the BEK results and the implications and this is discussed in section 6.

2. Zeta functions and Weyl series

We recall here how to calculate the Weyl series coefficients of the resolvent from the analytic continuation of associated spectral zeta functions (Voros 1992). Our starting point is the heat kernel of the spectrum defined by

$$\theta(t) = \sum_n \mathrm{e}^{-tE_n} \tag{7}$$

with the ansatz that, for the D-dimensional smooth systems considered here,

$$\theta(t) \sim \sum_{r=0}^{\infty} a_r t^{(r-D)/2} \qquad t \to 0^+. \tag{8}$$

The zeta function

$$\zeta(p) = \sum_n E_n^{-p} \tag{9}$$

is then related to the heat kernel by the Mellin transform

$$\zeta(p) = \frac{1}{\Gamma(p)} \int_0^\infty \mathrm{d}t\, t^{p-1}\theta(t) \qquad \mathrm{Re}\,(p) > -D/2. \tag{10}$$

Inserting (8) into (10) and separating the integration range into [0, 1] and $[1, \infty)$ we obtain

$$\zeta(p) \sim \frac{1}{\Gamma(p)} \left\{ \sum_{r=0}^{\infty} \frac{a_r}{p + (r-D)/2} + h(p) \right\} \tag{11}$$

where $h(p)$ is analytic in p (due to the exponential convergence of $\theta(t)$ for large t). It then follows that the a_r are given by Lesduarte and Romeo (1994):

$$\begin{aligned} a_r &= \Gamma\left(\frac{D-r}{2}\right) \mathrm{Res}\left\{\zeta(p),\, p = \frac{D-r}{2}\right\} \qquad 0 \leqslant r \leqslant D-1 \\ a_{2m+D} &= (-1)^m \zeta(-m)/m! \qquad m \in N \\ a_{2m+1+D} &= \Gamma(-m - \tfrac{1}{2}) \mathrm{Res}\{\zeta(p),\, p = -m - \tfrac{1}{2}\} \qquad m \in N. \end{aligned} \tag{12}$$

Finally to tie in with the Weyl coefficients of the resolvent function dealt with in BH, HT (up to a shift in index) we note the formal relationship between the regularized resolvent $g(s)$ and the heat kernel (BH equation (20)),

$$g(s) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-s^2 t} \left\{ \theta(t) - \sum_{r=0}^{D-2} a_r t^{(r-D)/2} \right\} \sim \sum_{r=D-1}^{\infty} \frac{c_r}{s^{r-D+2}}. \tag{13}$$

The Weyl terms of the regularized resolvent are thus

$$c_r = \Gamma\left(\frac{r-D}{2} + 1\right) a_r \qquad r \geqslant D-1 \tag{14}$$

which in turn can be determined from (12) as

$$\begin{aligned} c_{D-1} &= \pi \mathrm{Res}\{\zeta(p),\, p = \tfrac{1}{2}\} \\ c_{2m+D} &= (-1)^m \zeta(-m) \qquad m \in N \\ c_{2m+1+D} &= \pi(-1)^{m+1} \mathrm{Res}\{\zeta(p),\, p = -m - \tfrac{1}{2}\} \qquad m \in N. \end{aligned} \tag{15}$$

(Note that in reference HT the c-coefficient indices r ran from 1 whereas here they start at $D-1$, so that $c_1^{\mathrm{HT}} = c_{D-1}^{\mathrm{Here}}$.)

Thus we need to continue analytically the zeta function to the negative real axis, to calculate the Weyl series. The method of BEK achieves this, with some technical differences induced by the flux line. For that reason we now briefly sketch the unfluxed method.

3. The method of Bordag *et al*

The starting point is the Dirichlet representation of a (mass M particle) zeta function

$$\zeta(p) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{d_l}{(E_{nl} + M^2)^p} \tag{16}$$

where d_l is degeneracy of the eigenvalues E_{nl}. For any system with spherical symmetry such that the radial eigenvalues are Bessel functions, this sum can be represented as an integral

$$\zeta(p) = \sum_{l=0}^{\infty} \frac{d_l}{2\pi \mathrm{i}} \int_\gamma \frac{\mathrm{d}k}{(k^2+M^2)^p} \frac{\partial}{\partial k}\{\log J_{\nu(l)}(kR)\} \tag{17}$$

where $\nu(l)$ labels the eigenmodes and the contour γ is a clockwise-oriented curve enclosing all real-k solutions of

$$J_\nu(kR) = 0. \tag{18}$$

The representation (17) is only valid for a range of $\mathrm{Re}\,(p) > 0$. We seek to continue it analytically to $\mathrm{Re}\,(p) < 0$. The method outlined below is valid for general ν and, as we shall show, can be adapted to our $D = 2$ example by setting $\nu = l + \alpha$ and $l + 1 - \alpha$ with $d_l = 1 \forall l$. For the $D = 3$ spherical AB billiard, we shall have to modify this algorithm slightly.

First we focus on an unfluxed sphere. We insert an inert $k^{-\nu}$ into the derivative of (17) to prevent contributions from the origin, and deform the integration contour to the imaginary k axis:

$$\begin{aligned}\zeta_\nu &\equiv \int_\gamma \frac{\mathrm{d}k}{(k^2+M^2)^p} \frac{\partial}{\partial k}\{k^{-\nu} \log J_\nu(kR)\} \\ &= \frac{\sin \pi p}{\pi} \int_M^\infty \frac{\mathrm{d}k}{(s^2-M^2)^p} \frac{\partial}{\partial s}\{\log I_\nu(sR)\} \qquad k = \mathrm{i}s.\end{aligned} \tag{19}$$

The modified Bessel function is then expanded as

$$I_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{\mathrm{e}^{\nu\eta}}{(1+z^2)^{1/4}} \sum_{r=0}^{\infty} \frac{u_r}{\nu^r} \tag{20}$$

where

$$\begin{aligned}&\eta = \sqrt{1+z^2} + \log\left[z \Big/ \left(1+\sqrt{1+z^2}\right)\right] \qquad t = 1 \Big/ \sqrt{1+z^2} \\ &u_{r+1}(t) = \tfrac{1}{2} t^2(1-t^2)\, u_r'(t) + \tfrac{1}{8} \int_0^t \mathrm{d}\tau (1-5\tau^2)\, u_r(\tau) \qquad u_0(t) = 1.\end{aligned} \tag{21}$$

Then we define formally the expansion

$$\log\left[\sum_{r=0}^{\infty} \frac{u_r(t)}{\nu^r}\right] \sim \sum_{n=1}^{\infty} \frac{D_n(t)}{\nu^n}. \tag{22}$$

By simultaneously subtracting and adding N terms of this asymptotic expansion for $\nu \to \infty$ we can derive an expression for ζ_ν valid on the strip $(1-N)/2 < \mathrm{Re}\,(p) < 1$. The result is

$$\begin{aligned}\zeta_\nu &= Z_\nu(N) + \sum_{j=-1}^{N} A_j^\nu(p) \\ A_{-1}^\nu &= \frac{\sin \pi p}{\pi} \int_{MR/\nu}^\infty \mathrm{d}z \left[\left(\frac{z\nu}{R}\right)^2 - M^2\right]^{-p} \frac{\partial}{\partial z} \log(z^{-\nu} \mathrm{e}^{\nu\eta}) \\ A_0^\nu &= \frac{\sin \pi p}{\pi} \int_{MR/\nu}^\infty \mathrm{d}z \left[\left(\frac{z\nu}{R}\right)^2 - M^2\right]^{-p} \frac{\partial}{\partial z} \log(1+z^2)^{-1/4} \\ A_j^\nu &= \frac{\sin \pi p}{\pi} \int_{MR/\nu}^\infty \mathrm{d}z \left[\left(\frac{z\nu}{R}\right)^2 - M^2\right]^{-p} \frac{\partial}{\partial z} \left(\frac{D_j(t)}{\nu^j}\right)\end{aligned} \tag{23}$$

where $Z_\nu(N)$ is both analytic on the strip $(1-N)/2 < \mathrm{Re}\,(p) < 1$ and is zero for negative integer p: it therefore plays no role in calculating the c_n (cf (15)) and we focus henceforth on the A-coefficients.

BEK recognize the A-coefficients as ${}_2F_1$ hypergeometric functions. Representing these using Mellin–Barnes integrals it is possible to exchange the degeneracy sum and integral after a careful consideration of the contour.

After a somewhat lengthy calculation, the contribution of the A-coefficients to the full, zeta function can be recast as

$$A_j(p) = \sum_{l=0}^{\infty} d_l A_j^{\nu(l)} \tag{24}$$

where

$$\begin{aligned} A_{-1}(p) &= \frac{R^{2p}}{2\sqrt{\pi}\Gamma(p)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (MR)^{2j} \frac{\Gamma(j+p-1/2)}{p+j} \zeta_{\mathrm{H}}(2j+2p-2; 1/2) \\ A_0(p) &= -\frac{R^{2p}}{2\Gamma(p)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (MR)^{2j} \Gamma(j+p) \zeta_{\mathrm{H}}(2j+2p-1; 1/2) \\ A_i(p) &= -\frac{R^{2p}}{\Gamma(p)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (MR)^{2j} \zeta_{\mathrm{H}}(2-1+\mathrm{i}+2j+2p; 1/2) \\ &\qquad \times \sum_{a=0}^{i} x_{i,a} \frac{(\mathrm{i}+2a)\Gamma(p+a+j+\mathrm{i}/2)}{\Gamma(1+a+\mathrm{i}/2)}. \end{aligned} \tag{25}$$

Here ζ_{H} denotes a Hurwitz zeta function defined by

$$\zeta_{\mathrm{H}}(p; q) = \sum_{l=0}^{\infty} (l+q)^{-p} \qquad \mathrm{Re}\,(p) > 1 \tag{26}$$

and the $x_{i,a}$ are the coefficients of t in the Bessel-expansion polynomials (22):

$$D_i(t) = \sum_{a=0}^{i} x_{i,a} t^{\mathrm{i}+2a}. \tag{27}$$

The $x_{i,a}$ satisfy the recurrence relation given in appendix A of BEK (up to an overall multiplicative factor of -1 in the first equation). Note that for the unfluxed ball in $D = 3$, the degeneracy factor is $d_l = 2l + 1, \forall n$. As the mass M contributes only an overall $\exp(-M^2 t)$ prefactor to the heat kernel we may now set $M = 0$ without loss of generality, simplifying the subsequent analysis.

From (15) we require the zeta function at $p = -m$ and $p = -m - 1/2$. We can now fix the upper limit N of the subtracted sum in (25) so as to ensure that $Z_\nu(N)$ does not contribute. From the condition just below (23), we find that

$$N > \begin{cases} 2m+2 & p = -m - 1/2 \\ 2m+1 & p = -m \end{cases} \Rightarrow N = \begin{cases} 2m+3 & p = -m-1/2 \\ 2m+2 & p = -m. \end{cases} \tag{28}$$

We can thus represent the zeta functions as (cf (15))

$$\begin{aligned} \zeta(-m) &= \sum_{l=-1}^{2m+2} A_l(-m) \\ \mathrm{Res}\{\zeta(p), p = -1/2 - m\} &= \sum_{l=-1}^{2m+3} \mathrm{Res}\{A_l(p), p = -1/2 - m\}. \end{aligned} \tag{29}$$

Hence from (15), (2), (19), (23), (25) and (29) we find that

$$\begin{aligned} c_{D-1} &= \pi \operatorname{Res}\{\zeta(p), p = \tfrac{1}{2}\} \\ c_{2m+D} &= (-1)^m \sum_{l=-1}^{2m+1} A_l(-m) \qquad m \in N \\ c_{2m+1+D} &= \pi(-1)^{m+1} \sum_{l=-1}^{2m+2} \operatorname{Res}\{A_l(p), p = -1/2 - m\} \qquad m \in N. \end{aligned} \tag{30}$$

We are now in a position to examine how these coefficients are modified by the inclusion of flux.

4. $D = 2$

We consider a circular 2D billiard of radius R with Dirichlet boundary conditions and a single flux-line of strength α at the origin. The fluxed system still possesses axial spatial symmetry, but the time reversal symmetry is broken. The radial eigenfunctions are now non-integer-order Bessel functions

$$\psi_{nm}(r) = N_{nm} J_{|n-\alpha|}(j_{|n-\alpha|} r) \mathrm{e}^{\mathrm{i}m\phi} \qquad -\infty < n < +\infty \qquad m \geqslant 1. \tag{31}$$

The degeneracies of the eigenvalues $E_{np} = j^2_{|n-\alpha|,p}$ are also broken. We therefore adapt the BEK scheme by splitting the total zeta function into a sum of zeta functions each of non-degenerate eigenvalues. Assuming the BEK notation (19), we have for $0 \leqslant \alpha \leqslant 1/2$,

$$\begin{aligned} \zeta^{\text{Total}}_{D=2}(p) &= \zeta_{|\alpha|}(p) + \sum_{l=1}^{\infty} \{\zeta_{l-\alpha}(p) + \zeta_{l+\alpha}(p)\} \\ &= \sum_{l=0}^{\infty} \{\zeta_{l+1-\alpha}(p) + \zeta_{l+\alpha}(p)\} \\ &= \sum_{l=0}^{\infty} \{\zeta_{l+1/2+(1/2-\alpha)}(p) + \zeta_{l+1/2-(1/2-\alpha)}(p)\}. \end{aligned} \tag{32}$$

The last line demonstrates that $\zeta^{\text{Total}}_{D=2}(p)$ is symmetric in $(1/2 - \alpha)$. We can therefore calculate the $D = 2$ fluxed zeta function by summing non-degenerate BEK zeta contributions (17) with $\nu = l + 1 - \alpha$ and $\nu = l + \alpha$.

A careful study of the BEK method reveals that when allowing for the change in degeneracies the only effect is to modify the summed A-coefficients (25) as follows. In the case of a $D = 3$ unfluxed ball the degeneracy is $2l + 1$. The result of pulling back the Mellin contour to the left and subsequent sum is the appearance of Hurwitz zeta functions (cf BEK (3.15)–(3.18))

$$\sum_{l=0}^{\infty} (2l+1)(l+1/2)^{-(x+1)} = 2\sum_{l=0}^{\infty} (l+1/2)^{-x} = \zeta_{\mathrm{H}}(x; 1/2) \qquad x > 1. \tag{33}$$

Here x denotes a generic exponent ($\operatorname{Re} x > 1$). For the fluxed circle the non-degenerate zeta functions now involve the sums

$$\begin{aligned} \nu = l + 1 - \alpha : &\quad \sum_{l=0}^{\infty} (l + 1 - \alpha)^{-(x+1)} = \zeta_{\mathrm{H}}(x + 1; 1 - \alpha) \qquad x > 1 \\ \nu = l + \alpha : &\quad \sum_{l=0}^{\infty} (l + \alpha)^{-(x+1)} = \zeta_{\mathrm{H}}(x + 1; \alpha) \qquad x > 1 \end{aligned} \tag{34}$$

where the x take the same values as in the unfluxed sphere calculation. Thus the A-coefficients can be transposed from the unfluxed sphere calculation to the fluxed circle case by the substitution

$$\begin{aligned} \nu = l+1-\alpha: &\quad \zeta_{\mathrm{H}}(x;1/2) \to \tfrac{1}{2}\zeta_{\mathrm{H}}(x+1;1-\alpha) \\ \nu = \alpha: &\quad \zeta_{\mathrm{H}}(x;1/2) \to \tfrac{1}{2}\zeta_{\mathrm{H}}(x+1;\alpha). \end{aligned} \tag{35}$$

Summing the two non-degenerate contributions we therefore arrive at the following expressions:

$$\begin{aligned} A_{-1}(p) &= \frac{R^{2p}}{4\sqrt{\pi}}\frac{\Gamma(p-1/2)}{\Gamma(p+1)}\{\zeta_{\mathrm{H}}(2p-1;1-\alpha)+\zeta_{\mathrm{H}}(2p-1;\alpha)\} \\ A_0(p) &= -\frac{R^{2p}}{4}\{\zeta_{\mathrm{H}}(2p;1-\alpha)+\zeta_{\mathrm{H}}(2p;\alpha)\} \\ A_i(p) &= -\frac{R^{2p}}{\Gamma(p)}\{\zeta_{\mathrm{H}}(2p+j;1-\alpha)+\zeta_{\mathrm{H}}(2p+j;\alpha)\}\sum_{a=0}^{j} x_{j,a}\frac{(j+2a)\Gamma(p+a+j/2)}{\Gamma(1+a+j/2)}. \end{aligned} \tag{36}$$

From the general result (30), the Weyl coefficients are then

$$\begin{aligned} c_{2m+2} &= (-1)^m \sum_{l=-1}^{2m+2} A_l(-m) \qquad m \in N \\ c_{2m+3} &= \pi(-1)^{m+1}\sum_{l=-1}^{2m+3}\mathrm{Res}\{A_l(p),\, p=-1/2-m\} \qquad m \in N. \end{aligned} \tag{37}$$

We thus require the residues of the A-coefficients at $p = 1-n/2$ ($n = 0, 1$) and $p = -m-1/2$ and their values at $p = -m$ (m positive integers). For this we use the fact that the continuations of the Hurwitz zeta functions of order $-n$ (n positive integer) can be represented as Bernoulli polynomials:

$$\zeta_{\mathrm{H}}(-n;q) = -B_{n+1}(q)/(n+1). \tag{38}$$

After a tedious calculation, many cancellations take place due to the symmetries of the arguments of the ζ_{H} in the A-coefficients (36) and the corresponding properties of Bernoulli polynomials (Gradshteyn and Ryzhik 1994)

$$B_n(\alpha)+B_n(1-\alpha) = (1+(-1)^n)B_n(\alpha) = \begin{cases} 2B_n(\alpha) & n \text{ even} \\ 0 & n \text{ odd}. \end{cases} \tag{39}$$

We find that the only non-zero terms are

$$\mathrm{Res}\{A_0,\, p=-1/2-m\} = -\frac{R^{-1-2m}}{4}\delta_{m,-1} \tag{40}$$

where δ denotes the Kronecker delta and

$$\mathrm{Res}\{A_{2m+2},\, p=-1/2-m\} = -\frac{R^{-1-2m}}{\Gamma(-1/2-m)}\sum_{a=0}^{2m+2} x_{2m+2,a}\frac{\Gamma(a+1/2)}{\Gamma(a+m+1)} \qquad m \geqslant 0 \tag{41}$$

$$A_{-1}(-m) = -\delta_{m,0}\frac{\Gamma(-1/2)}{8\sqrt{\pi}}\{B_2(1-\alpha)+B_2(\alpha)\} = \left\{\frac{1}{12}-\frac{\alpha(1-\alpha)}{2}\right\}\delta_{m,0} \tag{42}$$

$$A_{2m+1}(-m) = (-1)^{m+1}m!R^{-2m}\sum_{a=0}^{2m+1}\frac{x_{2m+1,a}\Gamma(a+1/2)}{\Gamma(m+a+1/2)} \qquad m \geqslant 0. \tag{43}$$

Taking account of the vanishing values the formulae for the Weyl coefficients (37) become

$$\begin{aligned} c_{2m+2} &= (-1)^m A_{2m+1}(-m) \\ c_{2m+3} &= \pi(-1)^{m+1}\mathrm{Res}\{A_{2m+2}(p),\, p=-m-1/2\}. \end{aligned} \tag{44}$$

On inserting the non-zero values above, we obtain the apparently novel (cf BH table 1) and remarkably compact expressions

$$\begin{aligned} c_2 &= \sum_{l=0}^{1} A_{2l-1}(0) = \frac{1}{6} - \frac{\alpha(1-\alpha)}{2} \\ c_r &= -R^{2-r} \sum_{a=0}^{r-1} \frac{x_{r-1,a}\Gamma(a+1/2)\Gamma(r/2)}{\Gamma(a+r/2-1/2)} \qquad r \geqslant 3. \end{aligned} \tag{45}$$

Clearly c_2 agrees (as it should) with Berry's result (6). However, from (27) the Bessel expansion coefficients $x_{r,k}$ are independent of the flux α. Consequently the higher orders ($r > 3$) of the circular Weyl series are completely ignorant of the flux-line. Expression (45) for $r > 3$ is identical to the unfluxed case and so generates the coefficients found in table 1 of BH. We can therefore deduce that the higher orders of the 2D fluxed case behaves as (BH section 3)

$$c_r \sim \frac{1}{2\sqrt{2\pi}} \frac{\Gamma(r-1/2)}{4^r} \qquad r \to \infty. \tag{46}$$

Note that the $l = 4$ orbit that dominates here is not a twice-traversed radial diffractive orbit. This can be deduced from the constant shift term in the gamma function. From BH (4)–(7) we see that this constant is associated with the algebraic prefactor of the associated periodic orbit correction. A diffractive orbit's algebraic prefactor in the periodic orbit fluctuations is one half power down on a corresponding periodic orbit (Sieber (1999), equation (44)). Since the shift in (46) is the same as the unfluxed case, the $l = 4$ contribution is thus the classically forbidden diametrical periodic orbit of the unfluxed case.

The Weyl series is therefore almost totally insensitive to the periodic orbit arising from the diffractive flux line. This is somewhat surprising since the actions of all polygonal orbits encircling the flux line μ times acquire integer multiples of the Aharonov–Bohm flux.

$$sl \to sl \pm 2\pi \mathrm{i}\mu\alpha. \tag{47}$$

We postpone a discussion of this case until after the next section.

5. $D = 3$

For $D = 3$ we thread the flux line between the (geographical) poles in the interior of the ball of radius R and apply Dirichlet conditions on the (inner) surface. This converts the $SO(3)$ symmetry of the spherical billiard into a $U(1)$ axial symmetry.

The eigenvalues of the system considered here are (Elizalde *et al* 1993)

$$\begin{aligned} &E_{n,m,r} = j^2_{|m-\alpha|+n+1/2,r} \qquad m, \quad r \in \mathbb{Z} \quad n \in \mathbb{N} \\ &J_{|m-\alpha|+n+1/2}(j_{|m-\alpha|+n+1/2,r}R) = 0. \end{aligned} \tag{48}$$

Using equation (30) the Weyl series in $D = 3$ is given by

$$\begin{aligned} c_2 &= \pi \mathrm{Res}\{\zeta(p), p = 1/2\} \\ c_{2m+3} &= (-1)^m \zeta(-m) \qquad m \geqslant 0 \\ c_{2m+4} &= \pi(-1)^{m+1} \mathrm{Res}\{\zeta(p), p = -m-1/2\} \qquad m \geqslant 0. \end{aligned} \tag{49}$$

Again the flux splits the degeneracies and so the zeta function we wish to analytically continue is given by (19), (48) as

$$\zeta^{\mathrm{Total}}_{D=3}(p) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{+\infty} \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \mathrm{d}k\, (k^2 + M^2)^{-p} \frac{\partial}{\partial k} \ln\{J_{|m-\alpha|+n+1/2}(kR)\}. \tag{50}$$

The main difference between the unfluxed sphere in BEK and here is the form of the double sum in (50), cf (17). Both calculations are similar until the step (24), (25). The unfluxed case encounters sums in the A-coefficients (25) which can be represented as simple Hurwitz zeta functions (26). In the fluxed case these sums are more complicated, but can be cast in the form

$$f(p,\alpha) \equiv \sum_{m=-\infty}^{+\infty} \sum_{n=0}^{+\infty} \frac{1}{(|m-\alpha|+n+1/2)^p} \qquad \mathrm{Re}\,(p) > 1. \tag{51}$$

A careful partitioning of the sums allows us to evaluate these as

$$\begin{aligned} f(p,\alpha) = {} & \zeta_{\mathrm{H}}(p-1; 1/2-\alpha) - (1/2-\alpha)\zeta_{\mathrm{H}}(p; 1/2-\alpha) \\ & +\zeta_{\mathrm{H}}(p-1; 1/2+\alpha) + (1/2-\alpha)\zeta_{\mathrm{H}}(p; 1/2+\alpha). \end{aligned} \tag{52}$$

Note that when $\alpha = 0$ we have $f(p,0) = 2\zeta_{\mathrm{H}}(p-1; 1/2)$, in accordance with the unfluxed degenerate formulae of BEK. We can thus read off the A-coefficients from the unfluxed $D = 3$ case by replacing each of the Hurwitz zeta functions in BEK according to the rule

$$\zeta_{\mathrm{H}}(x, 1/2)|_{\mathrm{BEK}} \to \frac{1}{2}\left\{ \begin{array}{l} \zeta_{\mathrm{H}}(x, 1/2-\alpha) - (1/2-\alpha)\zeta_{\mathrm{H}}(x+1, 1/2-\alpha) \\ +\zeta_{\mathrm{H}}(x, 1/2+\alpha) + (1/2-\alpha)\zeta_{\mathrm{H}}(x+1, 1/2+\alpha) \end{array} \right\} = \tfrac{1}{2} f(x+1, \alpha). \tag{53}$$

In the analytic continuation of the zeta functions to the negative axis we observe the following simplifications for m zero or a positive integer (cf (38)) (Gradshteyn and Ryzhik 1994)

$$\begin{aligned} & f(-2m, \alpha) = 0 \\ & f(-2m+1, \alpha) = -\frac{B_{2m+2}(1/2-\alpha)}{2m+2} + (1/2-\alpha)\frac{B_{2m+1}(1/2-\alpha)}{2m+1}. \end{aligned} \tag{54}$$

The necessary values and residues can then be found. We omit the tedious details and state only the non-zero results:

$$A_0(-m) = -\frac{R^{-2m}}{2} f(-1-2m, \alpha) \tag{55}$$

$$A_{2m+2}(-m) = (-1)^{m+1} R^{-2m} m! \sum_{a=0}^{2m+2} \frac{x_{2m+2,a}\Gamma(1+a)}{\Gamma(a+m+1)} \tag{56}$$

$$A_{2n}(-m) = -2R^{-2m} m! f(-1+2n-2m, \alpha) \sum_{a=0}^{2n} \frac{x_{2n,a}(-1)^{a+n}}{\Gamma(a+n)(m-a-n)!} \qquad m > 3n \tag{57}$$

$$A_{2n}(-m) = -2R^{-2m} m! f(-1+2n-2m, \alpha) \sum_{a=0}^{m-n} \frac{x_{2n,a}(-1)^{a+n}}{\Gamma(a+n)(m-a-n)!} \qquad 3n \geqslant m \geqslant n. \tag{58}$$

The non-zero residues are then,

$$\mathrm{Res}\{A_{-1}(p); p = -1/2-m\} = \frac{R^{-1-2m}(-1)^{m+1} f(-3-2m, \alpha)}{2\sqrt{\pi}\,\Gamma(1/2-m)(m+1)!} \tag{59}$$

$$\mathrm{Res}\{A_{2m+3}(p); p = -1/2-m\} = -\frac{R^{-1-2m}}{\Gamma(-1/2-m)} \sum_{a=0}^{2m+3} \frac{x_{2m+3,a}\Gamma(1+a)}{\Gamma(a+m+3/2)} \tag{60}$$

$$\begin{aligned} \mathrm{Res}\{A_{2n-1}(p); p = -1/2-m\} = {} & \frac{2(-1)^m R^{-1-2m}}{\Gamma(-1/2-m)} f(-3+2n-2m, \alpha) \\ & \times \sum_{a=0}^{2n-1} \frac{x_{2n-1,a}(-1)^{a+n}}{\Gamma(a+n-1/2)(m-a-n+1)!} \qquad m \geqslant 3n-1 \end{aligned} \tag{61}$$

Table 1. Terms in the Weyl series of the fluxed 3D ball, expressed in terms of the flux parameter α.

$$c_3 = -\frac{1}{48} + \frac{\alpha}{4} - \frac{\alpha^2}{4}$$
$$c_4 R = -\frac{1}{315} + \frac{\alpha}{12} - \frac{\alpha^2}{24} - \frac{\alpha^3}{12} + \frac{\alpha^4}{24}$$
$$c_5 R^2 = -\frac{1}{960} + \frac{\alpha}{12} - \frac{\alpha^2}{24} - \frac{\alpha^3}{12} + \frac{\alpha^4}{24}$$
$$c_6 R^3 = -\frac{2}{3003} + \frac{2\alpha}{15} - \frac{7\alpha^2}{120} - \frac{7\alpha^3}{48} + \frac{\alpha^4}{16} + \frac{\alpha^5}{80} - \frac{\alpha^6}{240}$$
$$c_7 R^4 = -\frac{47}{80\,640} + \frac{17\alpha}{60} - \frac{13\alpha^2}{120} - \frac{\alpha^3}{3} + \frac{\alpha^4}{8} + \frac{\alpha^5}{20} - \frac{\alpha^6}{60}$$
$$c_8 R^5 = -\frac{3169}{4849\,845} + \frac{125\alpha}{168} - \frac{167\alpha^2}{672} - \frac{89\alpha^3}{96} + \frac{39\alpha^4}{128} + \frac{3\alpha^5}{16} - \frac{11\alpha^6}{192} - \frac{\alpha^7}{224} + \frac{\alpha^8}{896}$$
$$c_9 R^6 = -\frac{521}{593\,160} + \frac{487\alpha}{210} - \frac{571\alpha^2}{840} - \frac{145\alpha^3}{48} + \frac{7\alpha^4}{8} + \frac{59\alpha^5}{80} - \frac{49\alpha^6}{240} - \frac{\alpha^7}{28} + \frac{\alpha^8}{112}$$
$$c_{10} R^7 = -\frac{198\,641}{143\,416\,845} + \frac{1003\alpha}{120} - \frac{1039\alpha^2}{480} - \frac{6491\alpha^3}{576} + \frac{3335\alpha^4}{1152} + \frac{12\,019\alpha^5}{3840} - \frac{1001\alpha^6}{1280} - \frac{85\alpha^7}{384} + \frac{5\alpha^8}{96} + \frac{5\alpha^9}{2304} - \frac{\alpha^{10}}{2304}$$

$$\mathrm{Res}\{A_{2n-1}(p);\ p = -1/2 - m\} = \frac{2(-1)^m R^{-1-2m}}{\Gamma(-1/2-m)} f(-3+2n-2m, \alpha) \times \sum_{a=0}^{m-n+1} \frac{x_{2n-1,a}(-1)^{a+n}}{\Gamma(a+n-1/2)(m-a-n+1)!} \qquad 3n-2 \geqslant m \geqslant n-1. \tag{62}$$

Inserting these expressions into (30) we calculate the coefficients as

$$\begin{aligned} c_{2m+3} &= (-1)^m \sum_{n=0}^{m+1} A_{2n}(-m) \qquad m \geqslant 0 \\ c_{2m+4} &= (-1)^{m+1}\pi \sum_{n=0}^{m+2} \mathrm{Res}\{A_{2n-1},\ p = -m-1/2\} \qquad m \geqslant 0. \end{aligned} \tag{63}$$

The Weyl series can then be calculated from (54)–(63) using a symbolic algebra package.

We first choose to expand the c_r in terms of the flux parameter α. The results are displayed in table 1. In contrast to $D = 2$ we find that the flux does alter all the c_r, $r > 2$ in $D = 3$. The general form of the polynomials are

$$\begin{aligned} c_{2m+3} &= \sum_{p=0}^{m+1} c_{2m+3,2p} \frac{\alpha^{2p}}{R^{2m}} \\ c_{2m+4} &= \sum_{p=0}^{m+2} c_{2m+4,2p} \frac{\alpha^{2p}}{R^{2m+1}}. \end{aligned} \tag{64}$$

By comparison with the first column of table 1 HT, and allowing for the notational shift in index clearly when $\alpha = 0$ we recover the unfluxed $D = 3$ ball results. We assume the form of the unfluxed ball (noting the shift in r-index between HT and here)

$$c_r = A_0 \frac{\Gamma(r-1/2)}{l^r}(1 + o(1)) \tag{65}$$

and seek to estimate l, the dominating periodic orbit length in (2), for $\alpha \ll 1$. In figure 1 we plot $\log|c_r/\Gamma(r-1/2)|$ against r for different values of α. It is clear that the c_r exhibit a

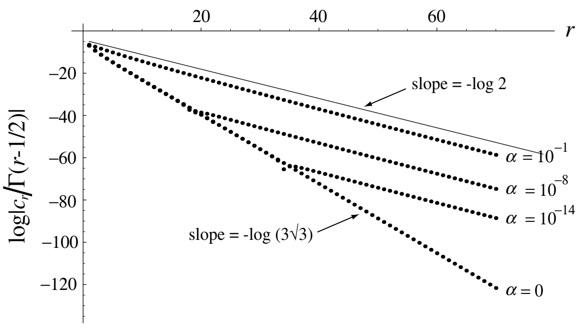

Figure 1. Plot of $\log|c_r/\Gamma(r-1/2)|$ against index r to estimate the dominant length l in the conjecture (65) as a function of the flux parameter α for the $D = 3$ ball.

transitional behaviour. For very small values of α the lower c_r are apparently dominated by the unfluxed result with $l = 3\sqrt{3}$, but the eventual dominant behaviour is $l = 2$, the latter being the diffractive orbit encountering the flux line. As α increases the $l = 2$ orbit rapidly asserts itself as the α-dependent terms dominate the constant, until by $\alpha \sim 0.1$ the influence of the $l = 3\sqrt{3}$ orbit is subdominant at this level of approximation. This is consistent with the conjectured form of the coefficient asymptotics (2), with the c_r undergoing an exchange of dominance between the $l = 3\sqrt{3}$ and 2 case as r and α vary.

To establish the form of the coefficients when the $l = 2$ dominates, we assume a form based on the evidence of figure 1

$$c_r = \frac{(r+\gamma)!}{l^r}\left\{a_0(\alpha) + \mathrm{O}\left(\frac{1}{r}\right)\right\} \tag{66}$$

where γ is a constant, independent of α. (Note the contributions from other periodic orbits $l_{\text{p.o.}} > l$ to (66) will be $\mathrm{O}(\exp\{-r\log(l/l_{\text{p.o.}})\})$ and so are negligible at this level of

Table 2. Terms in the Weyl series of the fluxed 3D ball, expressed in terms of the shifted flux parameter $\beta = \alpha - 1/2$.

$c_3 = \frac{1}{24} - \frac{\beta^2}{4}$
$c_4 R = \frac{817}{40\,320} - \frac{5\beta^2}{48} + \frac{\beta^4}{24}$
$c_5 R^2 = \frac{43}{1920} - \frac{5\beta^2}{48} + \frac{\beta^4}{24}$
$c_6 R^3 = \frac{115\,069}{3075\,072} - \frac{659\beta^2}{3840} + \frac{5\beta^4}{64} - \frac{\beta^6}{240}$
$c_7 R^4 = \frac{821}{10\,080} - \frac{359\beta^2}{960} + \frac{3\beta^4}{16} - \frac{\beta^6}{60}$
$c_8 R^5 = \frac{34\,538\,601\,043}{158\,919\,720\,960} - \frac{43\,175\beta^2}{43\,008} + \frac{557\beta^4}{1024} - \frac{25\beta^6}{384} + \frac{\beta^8}{896}$
$c_9 R^6 = \frac{3242\,537}{4730\,880} - \frac{42\,841\beta^2}{13\,440} + \frac{235\beta^4}{128} - \frac{4\beta^6}{15} + \frac{\beta^8}{112}$
$c_{10} R^7 = \frac{93\,726\,624\,813\,901}{37\,595\,865\,415\,680} - \frac{11\,470\,547\beta^2}{983\,040} + \frac{520\,061\beta^4}{73\,728} - \frac{12\,033\beta^6}{10\,240} + \frac{175\beta^8}{3072} - \frac{\beta^{10}}{2304}$

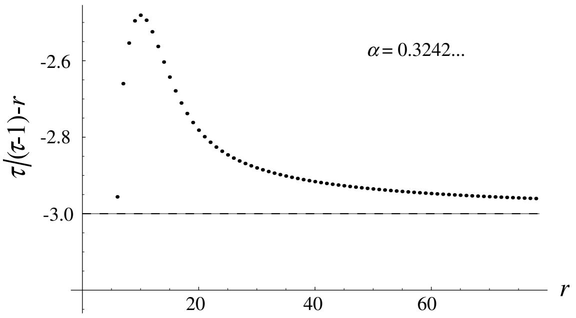

Figure 2. Plot of $\tau/(\tau - 1) - r$ (67) against index r to derive the estimate of the shift $\gamma \approx -3$ in the conjectured late term behaviour (66) for a random typical value of α for the $D = 3$ ball.

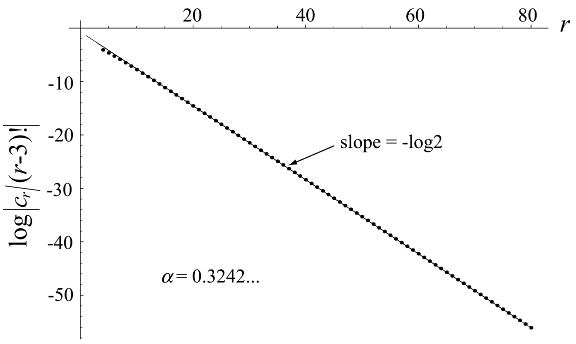

Figure 3. Plot of $\log|c_r/(r - 3)!|$ against index r to estimate the dominant length $l \approx 2$ in the conjectured late term behaviour (66) for a random typical value of α for the $D = 3$ ball.

approximation.) The value of γ can be found (HT 32) by considering the combination

$$\tau(r) \equiv \frac{c_r c_{r-2}}{c_{r-1}^2} \sim \frac{r+\gamma}{r+\gamma-1} \Rightarrow \gamma = \frac{\tau}{\tau-1} - r + \mathrm{O}\left(\frac{1}{r}\right) \tag{67}$$

and plotting the function $-r + \tau/(\tau - 1)$ against r. The dominant value of $l = 2$ can then be rechecked from a plot of $\log|c_r/(r + \gamma)!| \sim r \log l - \log|a_0(\alpha)|$ using the correct shift in the gamma function. This is summarized in figures 2 and 3, where we deduce that $\gamma = -3$, and confirm the $l = 2$ dominance.

The spectral functions of the fluxed case are known to be symmetric about $\alpha - 1/2$ (modulo 1). In order to check if the c_r preserve this symmetry we recast them as functions of $\beta = \alpha - 1/2$. Table 2 confirms that they are indeed all even in β and that they simplify dramatically. In this representation the polynomials take the form

$$\begin{aligned} c_{2m+3} &= \sum_{p=0}^{m+1} c_{2m+3,2p} \frac{\beta^2 p}{R^{2m}} \\ c_{2m+4} &= \sum_{p=0}^{m+2} c_{2m+4,2p} \frac{\beta^2 p}{R^{2m+1}}. \end{aligned} \tag{68}$$

We can analyse the individual β-dependent coefficients $c_{r,2p}$ by isolating all those associated with the powers β^{2p} and observing how they behave as a function of r. Again for each $c_{r,2p}$ we assume that it behaves asymptotically as

$$c_{r,2p} = \frac{(r+\gamma_p)!}{l_p^r} \left\{ a_{0,p} + \frac{l_p a_{1,p}}{(r+\gamma_p)} + \frac{l_p^2 a_{2,p}}{(r+\gamma_p)(r+\gamma_p-1)} + \mathrm{O}\left(\frac{1}{r^3}\right) \right\} \tag{69}$$

and seek to determine the γ_p and $a_{0,p}$. The results for different (typical) values of p are shown in figure 4. They demonstrate that $\gamma_p = -3$ and $l_p = 2$ for all p with a better rate of asymptotic agreement with r at smaller values of p.

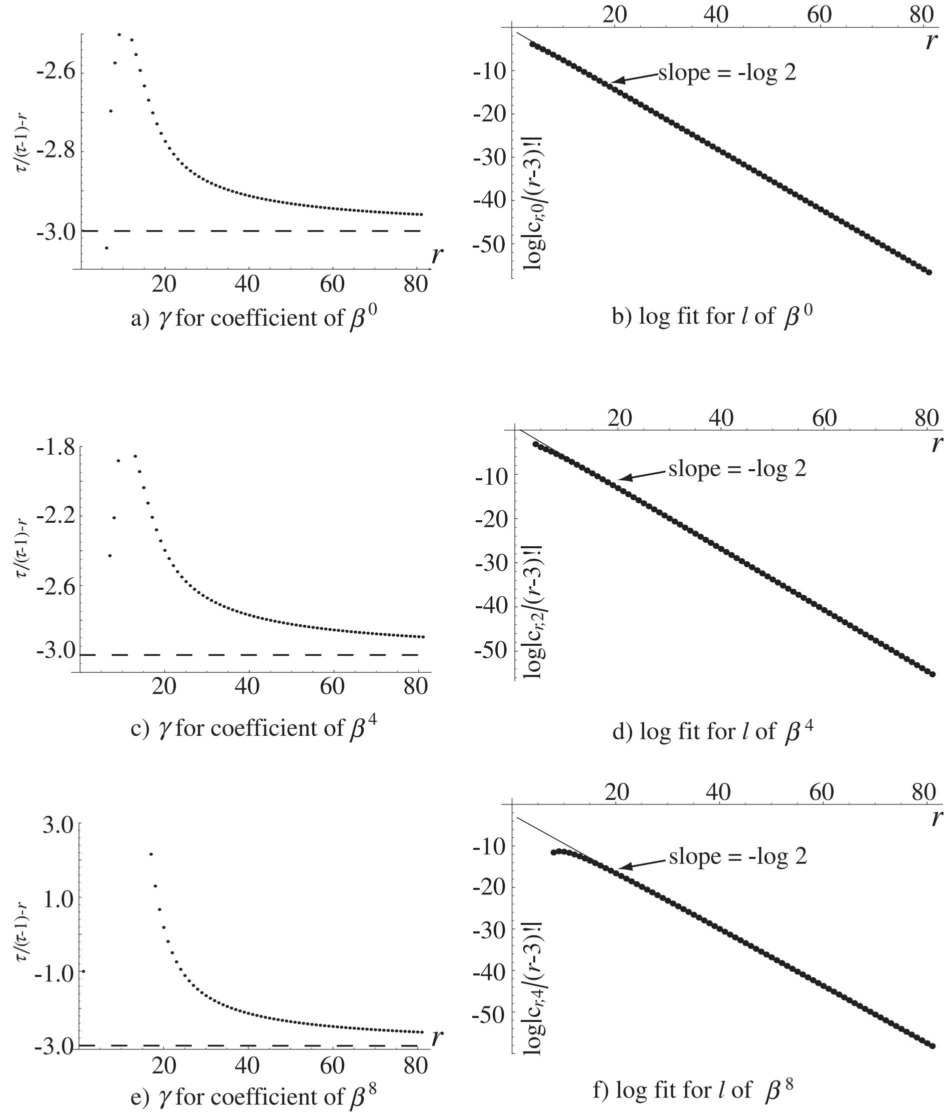

Figure 4. Plots of $\tau/(\tau-1)-r$ and $\log|c_{r,p}/(r-3)!|$ against index r to estimate the shifts $\gamma_p \approx -3$ and dominant lengths $l_p \approx 2$ in the coefficients of the β-powers in the conjectured late term behaviour (69) for the $D = 3$ ball for different values of the power.

Given these results and the form (69), we can use a Neville table algorithm (HT appendix C) to deduce the value of $a_{0,2p}$. Specifically, if

$$c_{r,2p} = \frac{(r-3)!}{2^{r-2}}\left\{a_{0,2p} + \frac{2a_{1,2p}}{(r-3)} + \mathrm{O}\left(\frac{1}{r^2}\right)\right\} \tag{70}$$

then if we define

$$\begin{aligned} S_{r,2p,1} &= \frac{2^{r-2}c_{r,2p}}{(r-3)!} \\ S_{r,2p,k} &= \frac{1}{(k-1)}\{(r-k+1)S_{r,2p,k-1} - (r-2k+2)S_{r-1,2p,k-1}\} \end{aligned} \tag{71}$$

Table 3. Scaled Neville table calculations of the terms within the coefficients of the Weyl series of the fluxed 3D ball, arranged in terms of powers of the shifted flux parameter, β^{2p} for a large value of $r = 35$. The higher iterations k in the table demonstrate convergence towards the estimate (73), with better convergence at low p.

$(-\mathrm{i}\pi)^{-2p}2\pi(2p)!S_{r,2p,k}$	$p = 0$	$p = 1$	$p = 2$	$p = 3$	$p = 7$
$k = 1$	0.954 083	0.942 343	0.884 086	0.784 588	0.250 717
$k = 5$	0.999 963	0.999 954	0.999 908	0.999 832	1.000 2
$k = 10$	0.999 998	0.999 992	0.999 953	0.999 811	0.993 737
$k = 15$	1.000 01	1.000 03	1.000 00	1.000 18	1.012 59

we have

$$S_{r,2p,k} = a_{0,2p} + \mathrm{O}\left(\frac{1}{r^k}\right). \tag{72}$$

It is possible to derive higher order approximations to $a_{1,2p}$, but here we only seek the leading order behaviour. We have also ignored the contribution of longer periodic orbits to the form (70), but these will contribute at $\mathrm{O}(1/2^r)$.

The results of this algorithm applied, 1, 5, 10 and 15 times for different values of p are displayed in table 3. By careful analysis for large r it is possible to deduce the trend

$$S_{r,2p,k} \sim \frac{(-1)^p\pi^{2p}}{2\pi(2p)!} \qquad k \gg 1 \tag{73}$$

with convergence better for lower values p, due to contamination from other periodic orbits (see HT).

As a consequence we can now assert that the $c_{r,2p}$ take the following form:

$$c_{r,2p} \sim \frac{(-1)^p\pi^{2p}}{2\pi(2p)!}\frac{(r-3)!}{2^{r-2}} \qquad r \to \infty \tag{74}$$

which suggests by comparison with (68) that

$$c_r \sim \frac{R}{2\pi}\frac{\Gamma(r-2)}{(2R)^{r-2}}\sum_{p=0}^{[r/2]}\frac{(\mathrm{i}\pi\beta)^{2p}}{(2p)!} \qquad r \to \infty \tag{75}$$

where $[r/2]$ denotes the integer part of $r/2$. Given that $-1/2 \leqslant \beta \leqslant 1/2$, the β-dependent sum can be further approximated as

$$\sum_{p=0}^{[r/2]}\frac{(\mathrm{i}\pi\beta)^{2p}}{(2p)!} \sim \cos(\pi\beta) = \sin(\pi\alpha) \qquad r \to \infty \tag{76}$$

to generate

$$c_r \sim \frac{R}{2\pi}\frac{\Gamma(r-2)}{(2R)^{r-2}}\sin(\pi\alpha) \qquad r \to \infty. \tag{77}$$

6. Discussion

It is possible to explain why the higher orders of the Weyl series of the circle are oblivious to the presence of the flux line by the following formal calculation.

Using the fluxed free Green function (A.1) and the analysis of BH, HT we can derive an expression for the boundary contributions to the resolvent as

$$g(s;\alpha) \equiv g_{\text{total}}(s;\alpha) - g_{\text{free}}(s;\alpha) = -\tfrac{1}{2}\sum_{m=-\infty}^{+\infty} f_{|m-\alpha|}(s) \tag{78}$$

where the $f_{|m-\alpha|}(s)$ is given by

$$f_m(s) = \left(1+\frac{m^2}{s^2}\right) I_m(s)K_m(s) - I'_m(s)K'_m(s) - \frac{I'_m(s)}{sI_m(s)}. \tag{79}$$

Note that the flux-altered term c_2 (6) is *not* included in (78) as the free Green function contribution has already been subtracted out. Using the Poisson sum,

$$\sum_{m=0}^{\infty} h_{m\pm\alpha}(s) = \sum_{\mu=-\infty}^{+\infty} \exp(\mp 2\pi \mathrm{i}\mu\alpha) \int_0^\infty \mathrm{d}m\, h_m(s) \exp(2\pi \mathrm{i} m\mu) \tag{80}$$

a short calculation results in

$$\begin{aligned} g(s;\alpha) &= -\int_0^\infty \mathrm{d}m\, f_m(s) - 2\sum_{\mu=1}^{\infty} \cos(2\pi\mu\alpha) \int_0^\infty \mathrm{d}m\, f_m(s)\cos(2\pi m\mu) \\ &= g(s;0) - 4\sum_{\mu=1}^{\infty} \sin^2(\pi\mu\alpha) \int_0^\infty \mathrm{d}m\, f_m(s)\cos(2\pi m\mu). \end{aligned} \tag{81}$$

The additional flux-induced asymptotic contributions therefore arise from the critical points of the extra m-integral for $s \to \infty$. Its existence is entirely due to the symmetry breaking. From the analysis of BH section 3 or HT section 4 we see that we must take into account the fact that $f_m(s)$ itself undergoes a Stokes phenomenon as the complex energy s is rotated back to the real energy axis. This gives rise to two types of contributions: those from algebraic terms in the asymptotic expansion of $f_m(s)$ and those from the resurgent exponential behaviour induced by a Stokes phenomenon (cf BH 60).

The exponential terms generate periodic orbit corrections from the saddle/end-points of the m integral. Except for the diffractive orbit, the lengths of the periodic orbits are identical to the unfluxed circle, although the actions do acquire the extra Aharonov–Bohm phases which lead to the $\sin^2 \pi\mu\alpha$ prefactor. When the corresponding contributions from $g(s;0)$ are added, the net result is a sequence of periodic orbit corrections, each with the appropriate symmetry-breaking prefactor, as found in Creagh (1996), Reimann *et al* (1996), Sieber (1999). The diffractive orbit arises from the endpoint contribution at $m = 0$.

The algebraic terms in the expansion of $f_m(s)$ result in a sum of Fourier integrals, each with only a single (real) critical point at $m = 0$. It is the expansion of each of these integrals about this endpoint that should lead to the flux contributions to the Weyl series.

Consideration of the form of the integral in (81) shows that the endpoint asymptotic expansion as $s \to \infty$ about this endpoint is given by equations (20), (26) of HT with $\nu = 0$

$$\begin{aligned} \int_0^\infty \mathrm{d}m\, f_m(s)\cos(2\pi m\mu) &= \tfrac{1}{2}\int_0^\infty \mathrm{d}m\, f_m(s)[\exp(2\pi\mu \mathrm{i}m) + \exp(-2\pi\mu \mathrm{i}m)] \\ &\sim \sum_{r=1}^{\infty}\sum_{k=0}^{r-1}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi\mu)^{2n}} \frac{q_{r,2k}}{s^{r+2n}} \left(\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}} \frac{x^{2k}}{(1+x^2)^{3r/2+1/2}} \right)_{x=0} \end{aligned} \tag{82}$$

for constants $q_{r,2k}$. Note that we have used the Debye expansions of the modified Bessel functions (Abramowitz and Stegun 1972, sections 9.7.7–9.7.10) since the integrals arising from the corresponding fixed expansions with m bounded, $s \to \infty$ (Abramowitz and Stegun 1972, sections 9.7.1–9.7.4) do not converge over an infinite range.

It is then clear that the odd derivatives of the even-in-x function in (82) vanish at the origin. Thus we have,

$$\int_0^\infty f_m(s)\cos(2\pi m\mu) = o(s^{-r}) \qquad \forall r. \tag{83}$$

Hence we can deduce that it is the asymptotic evenness of the functions $f_m(s)$ that result in the absence of any flux corrections to the Weyl series. This can be traced back to the evenness in $|m-\alpha|$ of the orders of the Bessel function eigenvalues.

The behaviour when $D = 3$ can be explained by recourse to the Green function of the system. Using the eigenfunctions (48) (Elizalde *et al* 1993) we can derive the Green function by standard techniques (e.g., Morse and Fesbach 1953) after recasting the generalized flux-dependent spherical harmonics in terms of Jacobi polynomials. After a detailed and very lengthy calculation, the resolvent can be expressed using the form (A.8) as

$$g(s;\alpha) = -\tfrac{1}{2}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{+\infty} f_{n+|m+\alpha|+1/2}(s) \tag{84}$$

which reduces to the unfluxed result of HT (14) when $\alpha = 0$. The above can be Poisson-summed to

$$g(s;\alpha) = -2\sum_{\mu=0}^{+\infty}(-1)^\mu\left\{\cos(2\pi\mu\alpha)(1-\tfrac{1}{2}\delta_{\mu,0})\int_0^\infty \mathrm{d}m\, m f_m(s)\cos(2\pi m\mu) + (\tfrac{1}{2}-\alpha)\sin(2\pi\mu\alpha)\int_0^\infty \mathrm{d}m\, f_m(s)\sin(2\pi m\mu)\right\} \tag{85}$$

where $\delta_{\mu,0}$ is the Kronecker delta. Following the procedure of expansion of (82) we could have used (85) to derive the Weyl series, but as previous stated, this is somewhat cumbersome. However from (85) we can identify two separate reasons for the presence of flux-dependent terms in the Weyl series.

First, the extra m in the cosine-dependent integral generates overall an odd integrand when the algebraic terms in the expansion of $f_m(s)$ are integrated. By comparison with (82) we see that the integrals will no longer then be $o(s^{-r})$ and so the α-dependent Weyl series contributions survive. These contributions persist when $\alpha = 0$ and so we can trace them to the interplay of the (odd) dimensionality of the billiard and the symmetry breaking, as noted in the case of unfluxed balls studied in HT.

The asymptotic contributions, algebraic in s, of the second integral in (85) do not vanish for $\alpha \neq 0$ due to the (odd-in-m) sine-term in the integrand (cf (82)). These contributions depend solely on the presence of flux and vanish (smoothly) when $\alpha = 0$. These contributions can therefore be attributed entirely to the symmetry breaking.

In order to check the asymptotic behaviour of the $D = 3$ terms (75), we can evaluate the contribution of the diffractive orbit from a consideration of the Stokes phenomenon of the integrands of (85), in the same vein as BH section 3 and HT section 4. This calculation reveals the asymptotic exponential contributions to be given by

$$g_{\mathrm{exp}}(s;\alpha) \sim -2\sum_{\mu=-\infty}^{+\infty}\sum_{p=1}^{+\infty} \mathrm{i}^p(-1)^\mu\left\{\cos(2\pi\mu\alpha)\int_0^\infty \mathrm{d}m\, m\frac{\sqrt{m^2+s^2}}{s^2} \times \exp(\pi\mathrm{i}m[p-2\mu]-pF(s,m)) + \mathrm{i}\left(\frac{1}{2}-\alpha\right)\sin(2\pi\mu\alpha)\int_0^\infty \mathrm{d}m\, \frac{\sqrt{m^2+s^2}}{s^2}\exp(\pi\mathrm{i}m[p-2\mu]-pF(s,m))\right\} \tag{86}$$

where $s = \mathrm{i}\sqrt{E}$, $\mathrm{Re}\,(E) > 0$ and

$$F(s, m) = 2\sqrt{m^2 + s^2} + 2m \log\left\{\frac{s}{m + \sqrt{m^2 + s^2}}\right\}. \tag{87}$$

We recall from BH that after a stationary phase approximation for large s, the summation indices p, μ respectively translate to the number of bounces on the boundary and the number of complete circulations of the origin of a classical orbit. Thus, following (Sieber 1999) we focus on the series of 1 bounce orbits and extract the terms with $p = 1$. The diffractive orbits then arise from the stationary phase linear endpoint contributions at the lower limit of integration

$$\begin{aligned} g_{\mathrm{diff}}(s;\alpha) &\sim -2\mathrm{i}\exp(-2s) \sum_{\mu=-\infty}^{+\infty} (-1)^{\mu} \Bigg\{ s\cos(2\pi\mu\alpha) \int_0^{\infty} \mathrm{d}x\, x \exp(sx\pi\mathrm{i}(1-2\mu)) \\ &\quad + \mathrm{i}(\tfrac{1}{2} - \alpha)\sin(2\pi\mu\alpha) \int_0^{\infty} \mathrm{d}x\, \exp(sx\pi\mathrm{i}(1-2\mu)) \Bigg\} \\ &\sim -2\mathrm{i}\exp(-2s) \sum_{\mu=-\infty}^{+\infty} (-1)^{\mu} \Bigg\{ s\cos(2\pi\mu\alpha) \left(\frac{\mathrm{i}}{s\pi(1-2\mu)}\right)^2 \\ &\quad + \mathrm{i}\left(\frac{1}{2} - \alpha\right) \sin(2\pi\mu\alpha) \left(\frac{\mathrm{i}}{s\pi(1-2\mu)}\right) \Bigg\}. \end{aligned} \tag{88}$$

If we now assume that the diffractive orbit does not encircle the flux line, then we must set $\mu = 0$. This leads to the conclusion that $g_{\mathrm{diff}} \sim 2\mathrm{i}\exp(-2s)/(s\pi^2)$, independent of α. From this argument we should expect a contribution from a periodic orbit of length 2 to be present in the orbit correction terms in the unfluxed case. This we know to be false, so the conclusion is that we must sum over all μ. The diffractive orbits thus consist of a radial segment from the boundary to the flux line, followed by μ encirclements of infinitesimal radius of the flux line, followed by a return radial segment to the boundary.

The μ-sums in (88) require the following evaluations (Gradshteyn and Ryzhik 1994)

$$\begin{aligned} \sum_{\mu=-\infty}^{+\infty} (-1)^{\mu} \frac{\cos(2\pi\mu\alpha)}{(1-2\mu)^2} &= 2\sin(\pi\alpha) \left\{ \begin{array}{ll} \frac{\pi^2\alpha}{4} & -\frac{1}{2} \leqslant \alpha \leqslant +\frac{1}{2} \\ \frac{\pi^2(1-\alpha)}{4} & +\frac{1}{2} \leqslant \alpha \leqslant +\frac{3}{2} \end{array} \right\} \\ &= \frac{\pi^2 \sin(\pi\alpha)}{2} \left(\frac{1}{2} - \left| \alpha - \frac{1}{2} \right| \right) \qquad 0 \leqslant \alpha \leqslant 1 \end{aligned} \tag{89}$$

$$\begin{aligned} \left(\frac{1}{2} - \alpha\right) \sum_{\mu=1}^{+\infty} (-1)^{\mu} \frac{\mu\sin(2\pi\mu\alpha)}{\mu^2 - (1/2)^2} &= \left(\frac{1}{2} - \alpha\right) \left\{ \begin{array}{ll} -\frac{\pi}{2}\sin(\pi\alpha) & -\frac{1}{2} \leqslant \alpha \leqslant +\frac{1}{2} \\ \frac{\pi}{2}\sin(\pi[1-\alpha]) & +\frac{1}{2} \leqslant \alpha \leqslant +\frac{3}{2} \end{array} \right\} \\ &= -\left| \alpha - \frac{1}{2} \right| \frac{\pi}{2} \sin(\pi\alpha) \qquad 0 \leqslant \alpha \leqslant 1. \end{aligned} \tag{90}$$

Thus we finally deduce that

$$\begin{aligned} g_{\mathrm{diff}}(s) &\sim 2\frac{\mathrm{i}}{s}\exp(-2s) \left\{ \frac{\sin(\pi\alpha)}{2} \left[\frac{1}{2} - \left| \alpha - \frac{1}{2} \right| \right] + \left| \alpha - \frac{1}{2} \right| \frac{\sin(\pi\alpha)}{2} \right\} \\ &= \frac{\mathrm{i}}{2s} \exp(-2s) \sin(\pi\alpha) \end{aligned} \tag{91}$$

which clearly vanishes at $\alpha = 0$ (as it should). The $\sin(\pi\alpha)$ term again can be interpreted as the symmetry-breaking modulation factor (Creagh 1996). It is not unity, as the diffractive orbit has been interpreted as cycling the flux line μ times. The overall modulation factor is the sum total of the effect of all $-\infty < \mu < +\infty$ cycles via (89)–(91).

From (13), in $D = 3$ the tail of the Weyl series takes the form

$$\sum_{r=N}^{\infty} \frac{c_r}{s^{r-1}}. \tag{92}$$

Hence, from BH (3) (noting the different power of s in (92)) and BH (7), we observe that an asymptotic behaviour of

$$c_r \sim A(\alpha)\frac{(r+\beta)!}{l^r} \tag{93}$$

generates, via a Stokes phenomenon (switching from $-1/2$ for s negative imaginary to $+1/2$ s positive imaginary) an exponential of the form

$$\exp(-sl)(sl)^{\beta+2}\mathrm{i}\pi A(\alpha). \tag{94}$$

Consequently when (91) and (94) are compared we predict that $\beta = -3$, $l = 2$, and $A = \sin(\pi\alpha)/2\pi$. This agrees with the general approximate form of (77). A more detailed calculation using the observed polynomial form (76) produces an identical result: the polynomial in α just leads to additional terms which generate null contributions to the Stokes phenomenon.

Thus we arrive at the following conclusions. First, in the fluxed $D = 2$ circle the origin of the diffractive orbit contributions to the periodic orbit fluctuations *is* from a Stokes phenomenon associated with Weyl series (BH section 3), followed by the appropriate saddle/end-point approximation (BH or HT). However, any trace of this Stokes phenomenon is wiped away by the corresponding integration of the algebraic terms, since it generates only terms beyond-all-orders, and thus no extra contribution to the Weyl series. This intricate process can be traced to the shift in the index of the Bessel functions in the Green function (A.1). This behaviour is probably therefore peculiar to the circle fluxed at the centre in $D = 2$.

In the $D = 3$ ball, the flux-dependent corrections arise from two sources; one an interplay between the odd-dimensionality and symmetry breaking, the other a term explicitly linked to the symmetry breaking.

Due to the symmetric nature of the example studied, we expect the results of the circle to be rather special. However further work on more general shapes in 2D would be required to confirm this. A study of the effect of uniform magnetic fields might also merit some investigation to observe if the cyclotron orbits play a role (Tanaka *et al* 1996).

Appendix

Berry (1986) demonstrated that for a 2D circle with a flux $\boldsymbol{B} = 2\pi\delta(\boldsymbol{r})$ the leading order magnetic correction to the spectral counting function arises from the free Green function. It is thus expected to be universal for all smooth flux-threaded 2D billiards and it suffices to calculate the correction for the circular billiard. Adopting the complex energy convention of BH, the unbounded Green function in the presence of a flux line becomes

$$G_{\mathrm{free}}^{(\alpha\neq 0)}(\boldsymbol{r}, \boldsymbol{r}_0; s) = \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty} I_{|m-\alpha|}(sr_<)K_{|m-\alpha|}(sr_>)\exp\{\mathrm{i}m(\theta_1-\theta_2)\} \tag{A.1}$$

where $r_< = \min(|\boldsymbol{r}|, |\boldsymbol{r}_0|)$, $r_> = \max(|\boldsymbol{r}|, |\boldsymbol{r}_0|)$. We define $\Delta g_{\mathrm{total}}(s;\alpha)$ as the difference between the fluxed and unfluxed resolvent

$$\Delta g_{\mathrm{total}}(s;\alpha) = \Delta g_{\mathrm{free}}(s;\alpha) + \Delta g(s;\alpha). \tag{A.2}$$

Here we have set

$$\Delta g_{\rm free}(s;\alpha) \equiv g_{\rm free}(s;\alpha) - g_{\rm free}(s;0) \tag{A.3}$$

where $g_{\rm free}$ is the contribution to the resolvent arising from the free Green function (A.1).

In terms of the Green function and the circular billiard we then have

$$\begin{aligned}\Delta g_{\rm free}(s;\alpha) &= \int_B \mathrm{d}\boldsymbol{r} \lim_{r\to r_0} \left[G_{\rm free}^{(\alpha\neq 0)}(\boldsymbol{r},\boldsymbol{r}_0) - G_{\rm free}^{(\alpha=0)}(\boldsymbol{r},\boldsymbol{r}_0)\right] \\ &= \int_0^1 \mathrm{d}r\, r \sum_{m=-\infty}^{+\infty} \left[I_{|m-\alpha|}(sr)K_{|m-\alpha|}(sr) - I_{|m|}(sr)K_{|m|}(sr)\right].\end{aligned} \tag{A.4}$$

The m-sum can be processed via a Poisson summation

$$\sum_{m=-\infty}^{+\infty} F(m) = \sum_{\mu=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}m\, F(m)\exp(2\pi \mathrm{i}\mu m) \tag{A.5}$$

to obtain

$$\Delta g(s;\alpha) = -8\int_0^1 \mathrm{d}r\, r \sum_{\mu=1}^{+\infty} \sin^2(\mu\pi\alpha) \int_0^\infty \mathrm{d}m\, I_m(sr)K_m(sr)\cos(2\pi\mu m). \tag{A.6}$$

The r-integral can be found in Howls and Trasler (1998) equation (21), and assuming it can be performed term-wise, we obtain

$$\Delta g(s;\alpha) = -4\sum_{\mu=1}^{+\infty} \sin^2(\mu\pi\alpha) \int_0^\infty \mathrm{d}m \left[f_m(s) + \frac{I_m'(s)}{sI_m(s)} - \frac{|m|}{s^2}\right]\cos(2\pi\mu m) \tag{A.7}$$

where

$$f_m(s) = \left(1+\frac{m^2}{s^2}\right) I_m(s)K_m(s) - I_m'(s)K_m'(s) - \frac{I_m'(s)}{sI_m(s)}. \tag{A.8}$$

That part of the integrand in (A.7) containing $f_m(s)$ can be expected to lead to non-algebraic terms (see BH section 3) and so does not play a role in the Weyl series. The remainder can be treated using the results of Howls and Trasler (1998) equations (33)–(35). Asymptotically we have

$$\begin{aligned}\int_0^\infty \mathrm{d}m \left[\frac{I_m'(s)}{sI_m(s)} - \frac{|m|}{s^2}\right]\cos(2\pi\mu m) &\sim \frac{1}{s^2}\int_0^\infty \mathrm{d}m \left[\sqrt{m^2+s^2} - |m|\right]\cos(2\pi\mu m) \\ &= \int_0^\infty \mathrm{d}x \left[\sqrt{1+x^2} - x\right]\cos(2\pi\mu s x) \\ &= \frac{1}{(2\pi\mu s)^2} - \frac{K_1(2\pi\mu s)}{2\pi\mu s}.\end{aligned} \tag{A.9}$$

The $K_1(2\pi\mu s)$ term is exponentially small and so we are left with the Weyl contribution as

$$\Delta g_{\rm Weyl}(s;\alpha) = -4\sum_{\mu=1}^{+\infty} \frac{\sin^2(\mu\pi\alpha)}{(2\pi\mu s)^2} = \frac{-\alpha(1-\alpha)}{2s^2} \tag{A.10}$$

as originally derived by Berry (1986) for the spectral counting function. In our notation, the fluxed free Green function modifies the coefficient of c_2 so that

$$c_2^{(\alpha\neq 0)} = \frac{1}{6} - \frac{\alpha(1-\alpha)}{2}. \tag{A.11}$$

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